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A new method for determination of a wave-ray trace based on tsunami isochrones*

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Abstract. A new method for the tsunami wave ray construction is proposed. It is based on isochrones that are built with the use of the wave travel times to a tsunami source. The wave ray approximation is determined as a broken line passing through intersection points of identical time isochrones from two supplementary dot sources. The method was tested on different types of the bottom relief.

1. Methods for the wave ray construction

There are a few different methods for the wave rays determination in a non-homogeneous area. The first way is constructing a ray by a numerical solution to the differential equations [1]

$$\frac{dx}{dt} = \frac{\vec{p}}{n^2(x)}, \qquad \frac{d\vec{p}}{dt} = \nabla \ln n(x), \tag{1}$$

with the initial conditions

$$\vec{x}\,\Big|_{t=0} = x^0, \qquad \vec{p}\,\Big|_{t=0} = n(x^0)\cdot\vec{\nu}^0. \tag{2}$$

Here $n(\vec{x}) = 1/c(\vec{x})$ is an inverse variable to the tsunami wave velocity at a given point and $\vec{\nu}^0$ is the initial vector which gives the outgoing ray direction from the source point x^0. The wave propagation velocity is dependent on the ocean depth only

$$c(x,y) = \sqrt{gH(x,y)}. \tag{3}$$

So, in order to obtain a wave ray trajectory, it is necessary to know the depth-distribution function all around the area of interest. In the process of the numerical solution of differential problem (1)–(2), a wave-ray trace can significantly change the direction above a small underwater mountain or a bottom subsidence. However, such sea bottom structures are actually unable to affect the tsunami wave, whose length exceeds the horizontal size of this bottom roughness. In geophysics, this method is called the “shooting” method. Such an effect can be illustrated by the wave rays, which have been calculated in the Northern Pacific (Figure 1). The ray outgoing point was

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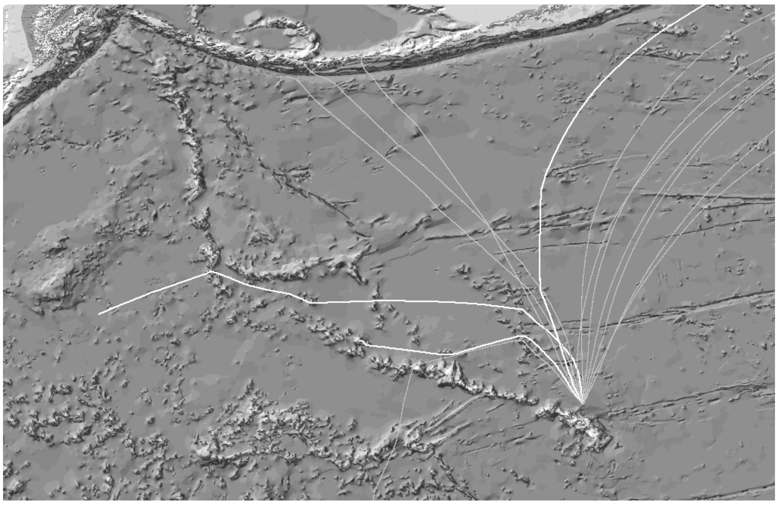

Figure 1. Wave rays in the North Pacific built by the "shooting" method

situated near to the Maui Island (Hawaii). The initial wave-ray direction was varying from the North-East to the North-West. It is seen that some ray-traces significantly change their direction when passing small bottom rises (see Figure 1).

As Figure 1 shows, some rays are crossing. For better illustration they are drawn as bold lines. In this paragraph, a wave ray is defined as a solution differential problem (1)–(2). Due to this, a ray crossing here is admissible. Such a method for the wave ray determination (construction) well works in domains with a model bottom relief, that allows us to study the tsunami behavior in different characteristic situations. For example, a wave-ray diagram allows us to visually explain the effect of the wave energy shoreward and seaward focusing in the case of tsunami source location over the continental slope [2]. Wave-ray traces from a round-shaped source, which is situated over a slope, are shown in Figure 2. Here the bottom slope is painted black, and the whole bottom profile is presented in the bottom part of the picture.

The second way to construct a wave ray emerging from a source is a consecutive advance of a wave-front line according to the eikonal equation [1]

$$|\nabla\tau|^2 = n^2(\vec{x}), \quad n(\vec{x}) = 1/c(\vec{x}), \tag{4}$$

with the initial condition

$$\tau(x, y, x_0, y_0) = 0, \tag{5}$$

where $c(x, y)$ is the wave propagation velocity at a point $\vec{x} = (x, y)$, and the equation $\tau(x, y) = t$ describes a wave front movement.

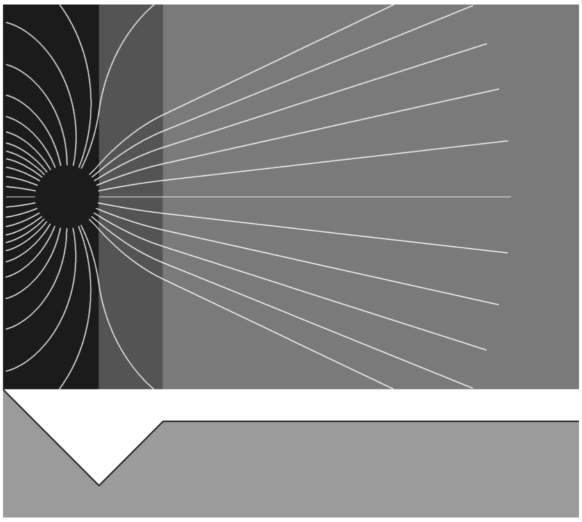

Figure 2. Wave-ray traces above the model bottom relief

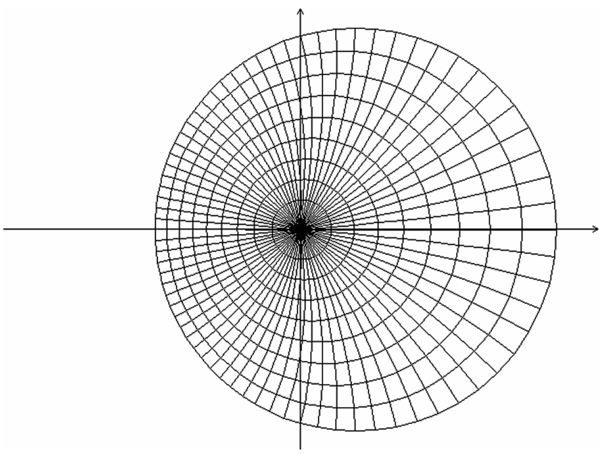

Figure 3. Tsunami wave fronts above the parabolic bottom

Through the numerical solution of the Cauchy problem (4)–(5) it is necessary to require the wave-front moving rate in the normal to a front line direction be equal to the wave propagation velocity at corresponding depth (see the Lagrange formula (3)). Here the concept of the wave ray means the optimal (fastest) route for disturbance propagation in non-homogeneous media where the wave propagation velocity varies. In the numerical realization of such an approach, the wave front is presented by the finite number of points. Tracing an advance of each of such points we obtain trajectories of wave rays [3]. An example of the ray construction in such a way is presented in Figure 3. Here a consecutive position of the wave front in the area where a depth increases from the left border to the right proportional to a squared distance to the left border is shown.

Another method to obtain a wave ray is based on the computation of wave travel times from the source origin point to all other points on a computational grid [5]. An approximate ray-trace path (the optimal wave propagation route) can be developed when calculating travel-times.

2. "Equitiming" method for the wave ray construction

The method proposed here is also based on calculation of tsunami travel times. Let there be required to build a wave ray leaving the source point (x_0, y_0) in a given direction. On a segment of the straight line passing through the point (x_0, y_0) and orthogonal to the ray emerging direction there are placed two additional dot sources (x_1, y_1) and (x_2, y_2). These new points must be located on the same distance from the source origin point (x_0, y_0). Tsunami travel-time charts (isochrones for fixed propagation time values) must be built for both these additional source locations. Tsunami isochrones can be obtained in various ways. The first way is to calculate consecutive positions of the wave front [3] (see Figure 3). Another way is to calculate tsunami travel times at all grid-points and then on their basis to construct a set of isolines. If these wave isochrones correspond to the same moments of time, then the broken line that passes isochrones crossing points will approximate the wave ray which has left the point (x_0, y_0) in the orthogonal direction to the segment connecting points (x_1, y_1) and (x_2, y_2). This algorithm is illustrated by Figure 4, where isochrone charts for the two dot sources (x_1, y_1) and (x_2, y_2) and the resulting wave ray are presented. At a smaller scale, the whole computational area is drawn in Figure 5.

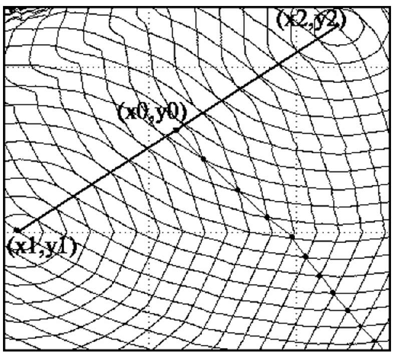

Figure 4. The scheme of wave ray determination using tsunami isochrones

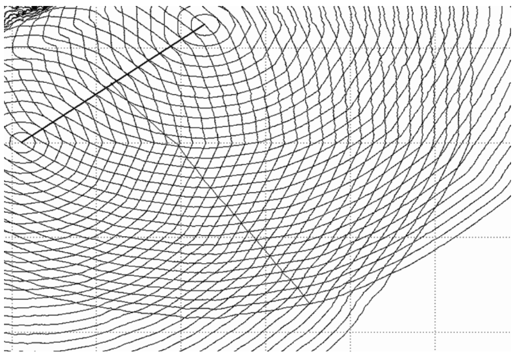

Figure 5. The whole computational area

Now, let us dwell on the tsunami isochrones construction. The simplest way is a numerical algorithm for finding consecutive positions of a wave front in a two-dimensional computational area [3]. In this case the wave-front points movement is calculated in the Cartesian coordinates (x, y). The gridded bathymetric data is recalculated at any spatial location (x, y) using the linear interpolation. Another way requires a preliminary calculation of travel times from these two additional dot tsunami sources (x_1, y_1) and (x_2, y_2) to all other points of the computational grid [5], and then the construction procedure based on this travel-times array of a set of isolines. It is necessary then to find points of the intersection of isolines corresponding to the identical moments of time.

Figure 6 presents tsunami isochrones from the two closely located dot sources (x_1, y_1) and (x_2, y_2) over the bottom slope. The grid step in the 1000×1000 computational domain was taken as 1000 m in both directions. In this case, the depth linearly increases from zero on the lower border to 2000 m on the top.

From the center of the segment connecting these two sources, the broken line passing through points of intersection of isolines is constructed. For

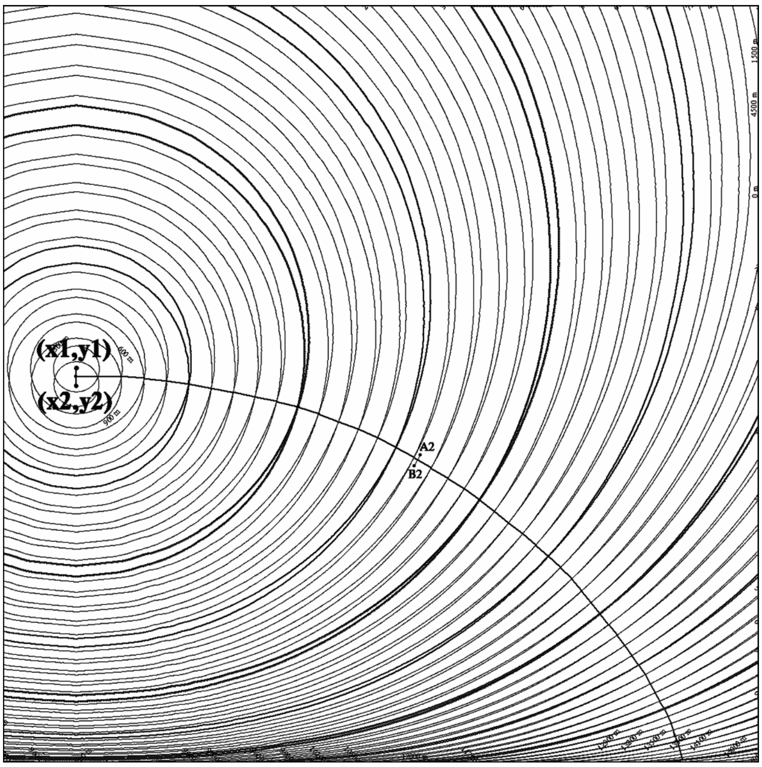

Figure 6

such a bathymetry, the exact shape of a wave ray representing a segment of a cycloid [4] is found. According to this exact solution, the ray, which at distance of 510 km off the coast was directed parallel to the coastline, will reach the coast (the lower border in Figure 6) approximately 801 km away from the the ray emerging point projection. Here the source was located 100 km off the left boundary. Hence, theoretically, a wave ray will come to the coast at a distance of 901 km from the left boundary that is well correlated with the ray obtained by the method offered (see Figure 6). A similar result turns out when constructing ray in an area with a constant depth. In this case the ray trajectory should represent a straight line, as has been confirmed by the numerical calculation.

It is necessary to notice that building isolines as it is, is not an easy task, which can be solved with application of additional programs. But it is possible to find the isochrones intersection points without building isolines. For this purpose, it is sufficient to calculate the wave travel times to all the grid-points of a computational area from each of these two dot sources (the two $N \times M$ numerical arrays $T_1(i,j)$ and $T_2(i,j)$, $i = 1,\ldots,N$, $j = 1,\ldots,M$). Then by subtraction of the corresponding elements of the arrays T_1 and T_2 it is necessary to create a numerical array $DT(i,j)$ of a travel-time difference at each grid-point. This array will contain negative, positive and (that is almost improbable) zero values. The isochrones intersection points can be found by the search for pairs of the neighboring points, where there is a transition from the negative value of $DT(i,j)$ to the positive one or vice versa. For example, if at the points (i,j) and $(i+1,j)$, the travel-time difference DT has a different sign, then a point of isochrones intersection (the point of a wave ray passage) must be located somewhere on a segment connecting these two points. A more precise position of this point can be found with allowance for the values $DT(i,j)$ and $DT(i+1,j)$, as well as for the depths at these two points. Thus, having looked through the whole computational domain in vertical and across, it is possible to find the location (to within one spatial step of a grid) of the tsunami isochrones intersection points. In Figure 7, the trajectory of the same wave ray which has been earlier constructed by means of isolines (see Figure 6) is presented. This ray trace coincides with the one found earlier. This proves that the proposed way of the search for isochrones intersection points is equivalent to a method required for constructing of tsunami travel-time field isolines.

Now let us prove that a broken line built in such a way is approximation of a wave-ray trace. If ray is sought for as an optimal route for the wave propagation from a source to a detector, such wave rays cannot be crossed among themselves, except in a source and in a receiver. So, let there be two dot sources A, B and the source C located between them. Let us take a look at the wave-ray trace approximation which has been built using the method proposed (the black dash line in Figure 8). Let the point D be an

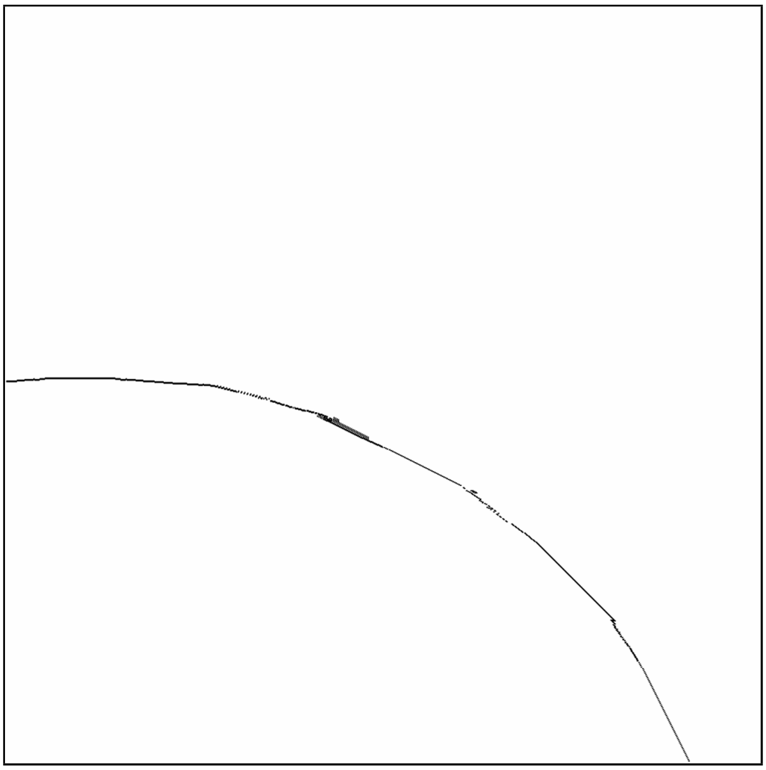

Figure 7

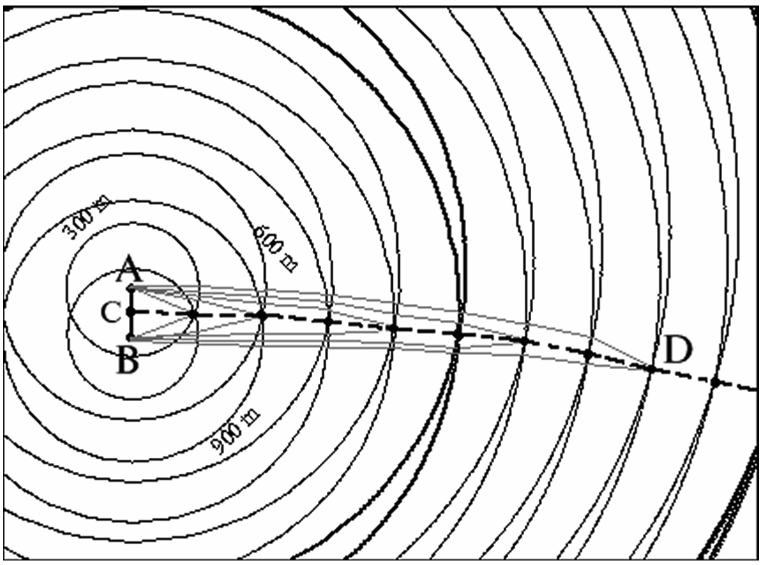

Figure 8

intersection point of 40-minutes wave isochrones from the sources A and B. Grey lines in Figure 8 present the wave rays connecting the point D with the sources A and B. As as been noticed, the wave ray connecting D and C cannot be crossed with the rays which connect the point D with A and B. Therefore this line will be located between this pair of wave rays (grey lines in Figure 8). As the ray CD is to be between the pair of the rays connecting points of the isochrones intersection every five minutes, it has no possibility to go away (to a considerable distance) the dash line passing through the isochrones intersection points. The actual wave ray during each moment of time is located in the ray tube formed by a pair of wave rays connecting a point of isochrones intersection with the sources A and B (grey lines in Figure 8). Thus, the dash line passing through the isochrones intersection points is approximation of a wave-ray trace starting from the point C, located in the middle of the segment AB, and orthogonal to it.

At a long distance from the point C, it can be difficult (almost impossible) to define the point of intersection of a pair of isolines, because they merge throughout some arch. Therefore, it is useful to offer the update of the method proposed excluding such a situation. Its essence consists in the following: at the first stage, we find the location of the isochones intersection points up to some moment of time. In Figure 9 representing a part of the whole computational area, resulted in Figure 6, this part of a wave-ray trace is drawn by a black line. Then on the last segment of this broken line we will set a new point C_2 for starting our ray continuation. For this purpose, at this point C_2 we build a segment, orthogonal to the last segment of the way-ray approximation constructed at the first stage. Let us set on it a new

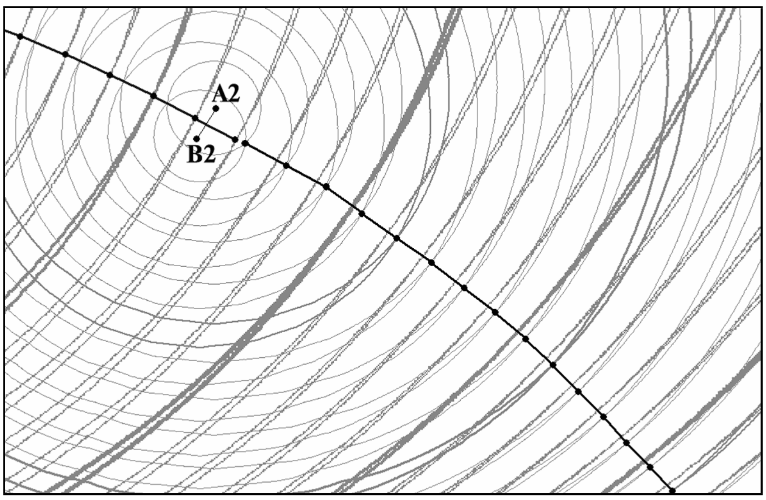

Figure 9

pair of supplemental dot sources A_2 and B_2 equally spaced from the point C_2 (see Figure 9). Then again we calculate the tsunami travel times from each of these two sources to other grid points (up to some new time limit). Thus, we repeat (as well as with the initial two dot sources (x_1, y_1) and (x_2, y_2)) the procedure of searching for subsequent points through which a wave ray passes. In Figure 9, it is clearly visible that at such a distance from the points (x_1, y_1) and (x_2, y_2) as the new points A_2 and B_2 are situated, it is difficult to uniquely define the intersection points location of tsunami isochrones propagating from sources (x_1, y_1) and (x_2, y_2).

At the same time, after setting additional tsunami sources A_2 and B_2, there are no problems with definition of further points of intersection of isochrones from these new source origins (the curves with smaller radii of the curvature in Figure 9). If a computational area is large enough (also, a wave ray has a big extent), such a procedure can be applied several times.

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